KNOX GRAMMAR SCHOOL

2021 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided as a separate document
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–10)

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 An examination consists of 20 questions, each having four possible answers. A student guesses the answer to every question. Let X be the number of correct answers. What is the value of $E(X)$?

A. 4

B. 5

C. 10

D. 16

2 Evaluate $|\underset{\sim}{a}+\underset{\sim}{b}|\underset{\sim}{c}$ given that $\underset{\sim}{a}=\begin{pmatrix}1\\2\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}3\\1\end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix}6\\-2\end{pmatrix}$.

A. 18

B. $\begin{pmatrix}30\\-10\end{pmatrix}$

C. 20

D. $5\sqrt{40}$

3 Given that $h(x)=\sqrt{2-x}$, what are the domain and range of $h^{-1}(x)$?

A. $x\le 2,\quad y\ge 0$

B. $x\le 2,\quad y\le 0$

C. $x\ge 0,\quad y\le 2$

D. $x\ge 0,\quad y\le -2$

4 Which of the following is a primitive of $\frac{4}{\sqrt{1-4x^2}}$?

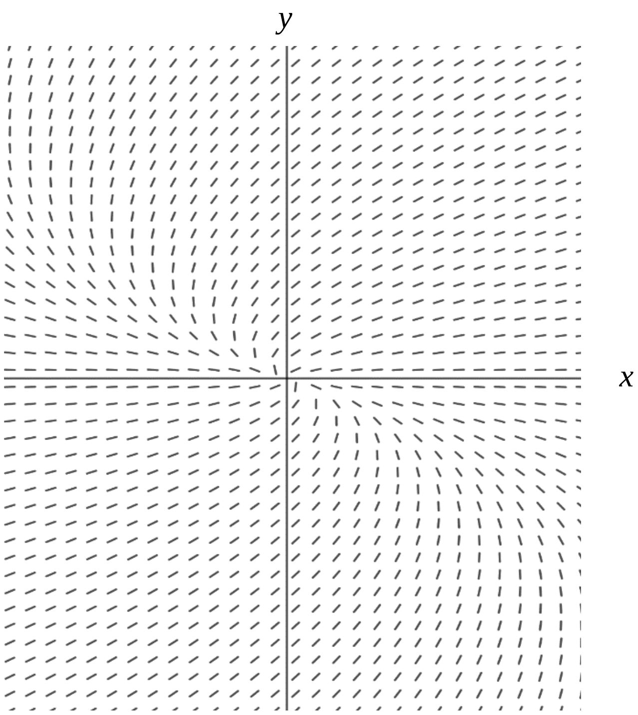
A. $2\sin^{-1}(2x)+C$

B. $2\sin^{-1}\left(\frac{x}{2}\right)+C$

C. $\sin^{-1}\frac{x}{2}+C$

D. $\sin^{-1}(2x)+C$

5 The slope field for a differential equation is shown below



Which of the following could be the differential equation represented?

A. $\frac{dy}{dx}=\frac{x}{x+y}$

B. $\frac{dy}{dx}=\frac{y}{x+y}$

C. $\frac{dy}{dx}=\frac{x}{x-y}$

D. $\frac{dy}{dx}=\frac{y}{x-y}$

6 Three candidates are running for election. There are 202 people who vote. The winner is the person who gets the most votes. What is the minimum number of votes needed for someone to win the election?

A. 101

B. 102

C. 67

D. 68

7 Which expression would be the correct answer to $\int \sin(5x)\sin(3x)\,dx$?

A. $\frac{1}{4}\cos(2x)+\frac{1}{16}\cos(8x)+C$

B. $\frac{1}{4}\cos(2x)-\frac{1}{16}\cos(8x)+C$

C. $\frac{1}{4}\sin(2x)+\frac{1}{16}\sin(8x)$ +C

D. $\frac{1}{4}\sin(2x)-\frac{1}{16}\sin(8x)+C$

8 A team of 7 is to be selected from 10 people and then sat around a circular table.
In how many ways can this be done?

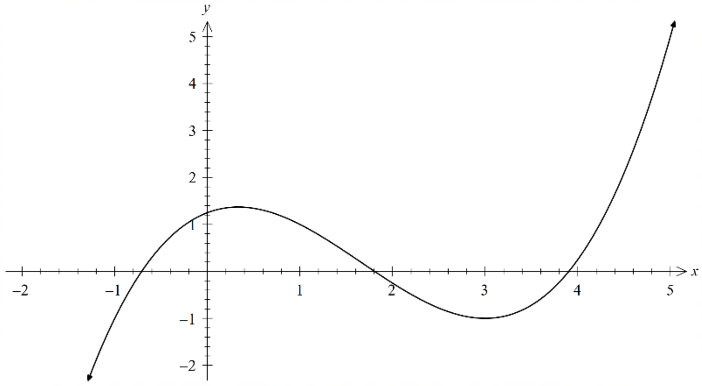
A. $\frac{10!}{7!3!}\times 7!$

B. $\frac{10!}{7!3!}\times\frac{1}{5!}$

C. $\frac{10!}{7\times 3!}$

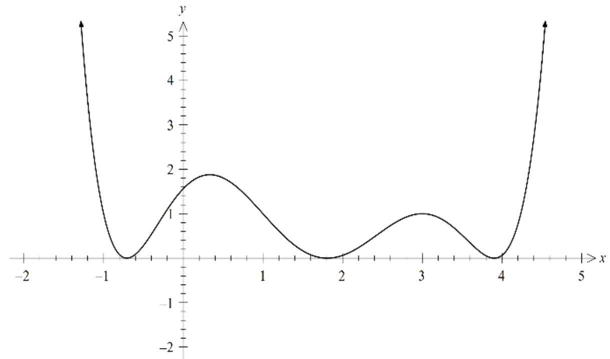
D. $\frac{10!}{7!3!}$

9 The graph of the function $y = f(x)$ is drawn below

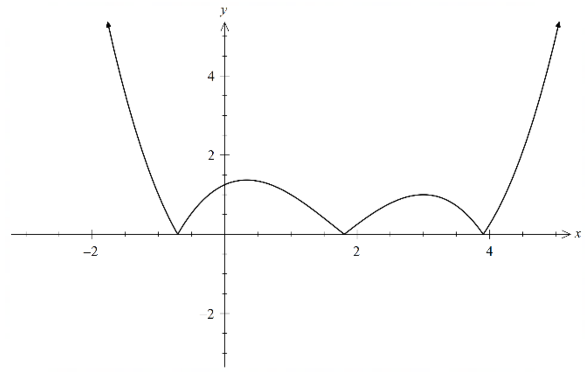


Which of the following represents the graph of $y = \left| f(|x|) \right|$?

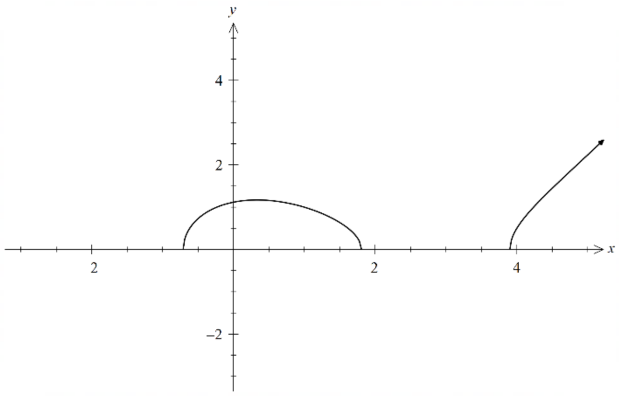
A



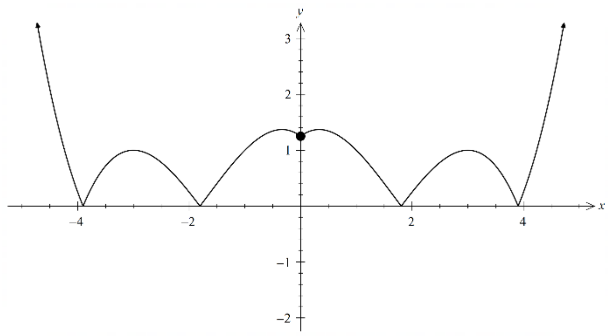
B



C



D



10 The parametric equations of a curve C are given by $x = \operatorname{cosec}^2 t$ and $y = \cos^2 t$.
What is the Cartesian equation of C ?

A. $y = 1 - \frac{1}{x}$

B. $y = 1 + \frac{1}{x}$

C. $xy = 1$

D. $y^2 = 1 - \frac{1}{x^2}$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

- Answer each question on the writing paper you have printed off.
- Start each question on a new page.
- For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) $P(x) = 2x^3 + 11x^2 + 12x - 9$.

(i) Show that $P(-3) = 0$. **1**

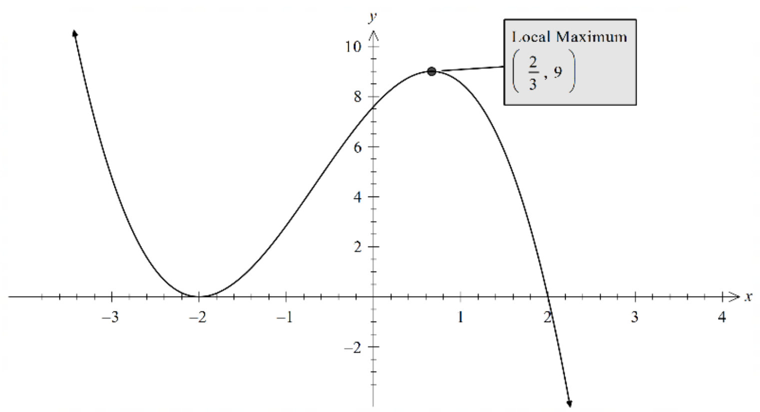
(ii) Show that $P'(-3) = 0$. **1**

(iii) Write $2x^3 + 11x^2 + 12x - 9$ as the product of 3 linear factors. **1**

(b) For what value(s) of a are the two distinct vectors $\begin{pmatrix} a \\ 2a+6 \end{pmatrix}$ and $\begin{pmatrix} a+1 \\ -1 \end{pmatrix}$ perpendicular? **2**

(c) The diagram shows the sketch of $y = f(x)$. Sketch the graph of $y^2 = f(x)$. **3**

Indicate the local maximum turning point on your diagram.



(d) By letting $t = \tan\frac{x}{2}$, solve $4\sin x - 3\cos x = 3$, for $0 \le x \le 2\pi$. **3**

(e) Solve $2\sin^2 2x - 1 = 0$, over the domain $[0, \pi]$. **2**

(f) A function is defined by $f(x) = \sin^{-1}(1-x)$.

(i) Find the domain of $f(x)$. **1**

(ii) Draw a neat sketch of $y = f(x)$. **1**

Question 12 (15marks) Start a new page.

(a) Use mathematical induction to prove that $n^3 + 2n$ is divisible by 3, for all positive integers n. **3**

(b) Weather observations in the town of Dampville have established that the probability of rain on any given day is 0.8.

Observations are made for 100 consecutive days.

Let X be the random variable representing the number of rainy days.

(i) Find the expected value $E(X)$. **1**

(ii) Find the standard deviation of X. **1**

(iii) By considering a normal distribution find the approximate probability that $76 \le X \le 84$. **1**

(c) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2x \, dx$. **3**

(d) Solve the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$, given that $y(-1) = 1$. **3**

Express your answer in the form $ay^3 + bx^2 + cy + d = 0$, where a,b,c,d are integers.

(e) The roots of $x^3 - 2x^2 + 5x + 1 = 0$ are α, β and γ. Find the value of

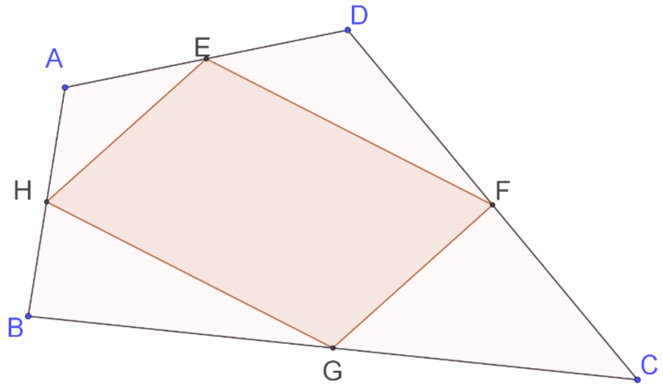
(i) $5\alpha + 5\beta + 5\gamma$. **1**

(ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

(iii) $\alpha^2 + \beta^2 + \gamma^2$. **1**

Question 13 (15marks) Start a new page.

(a) Varignon's theorem tells us that if you join the midpoints of the sides of any quadrilateral, the resulting quadrilateral will be a parallelogram.



E, *F*, *G* and *H* are the midpoints of *AD*, *DC*, *CB* and *BA* respectively.
Let $\overrightarrow{BA} = \underset{\sim}{u}$, $\overrightarrow{AD} = \underset{\sim}{v}$, $\overrightarrow{DC} = \underset{\sim}{w}$ and $\overrightarrow{CB} = \underset{\sim}{z}$.

(i) Explain why $\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w} + \underset{\sim}{z} = 0$. **1**

(ii) Show that $\overrightarrow{HE} = \frac{1}{2}\left(\underset{\sim}{u} + \underset{\sim}{v}\right)$. **1**

(iii) Show that $\overrightarrow{GF} = -\frac{1}{2}\left(\underset{\sim}{w} + \underset{\sim}{z}\right)$. **1**

(iv) Use your result in (i) to explain why *HE* is parallel to *GF* **2**

(b) Use the substitution $u = 3 + e^x$ to find the exact value of $\int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}}\,dx$. **3**

(c) A population of monkeys on a small island is an example of logistic growth.
The population of the monkeys *P* is given $P = \frac{50}{1+24e^{-2t}}$ and *t* is in years.

(i) Using this model, what is the carrying capacity of the population over time. **1**

(ii) Show that $\frac{dP}{dt} = \frac{1}{25}P\left(50 - P\right)$. **4**

(iii) Draw a neat sketch of $P = \frac{50}{1+24e^{-2t}}$, taking care to consider the shape of the function at $t = 0$. **2**

Question 14 (15 marks) Start a new page

(a) Find the value of n if ${}^{n}C_2 + {}^{n}C_1 + {}^{n}C_0 = 172$. **3**

(b) Let $f(x) = \dfrac{1}{\sin^{-1} x}$.

Find $f'(x)$ and the largest possible domain for which $f'(x)$ is defined. **3**

(c) (i) By considering the expansion of $(1+x)^{2n}$ show that. **4**

$$\binom{2n}{1} + 2\binom{2n}{2} + 3\binom{2n}{3} + \ldots\ldots + 2n\binom{2n}{2n} = n \times 2^{2n}.$$

(d) In a certain school, 23% of Year 12 students study Mathematics Extension 1.

(i) If X represents the number of students who study Mathematics Extension 1, describe the skewness of the binomial distribution for $P(X = x)$. **1**

You must give reasons for your answer.

(ii) The Principal meets with a random sample of 60 Year 12 students, to discuss their Year 12 HSC courses. **4**
What is the probability that more than 30% of the students that meet with the Principal study Mathematics Extension 1?
(You may assume that the sample of students is approximately normally distributed, and make reference to this extract below from a table of z scores).

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

End of Paper

KNOX GRAMMAR SCHOOL

2021 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1
SOLUTIONS

General Instructions

Reading time – 10 minutes

- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70
Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 An examination consists of 20 questions, each having four possible answers. A student guesses the answer to every question. Let X be the number of correct answers. What is the value of $E(X)$?

A. 4

B. 5

C. 10

D. 16

2 Evaluate $|\underset{\sim}{a}+\underset{\sim}{b}|\underset{\sim}{c}$ given that $\underset{\sim}{a}=\begin{pmatrix}1\\2\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}3\\1\end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix}6\\-2\end{pmatrix}$.

A. 18

B. $\begin{pmatrix}30\\-10\end{pmatrix}$

C. 20

D. $5\sqrt{40}$

3 Given that $h(x)=\sqrt{2-x}$, what are the domain and range of $h^{-1}(x)$?

A. $x \le 2, \quad y \ge 0$

B. $x \le 2, \quad y \le 0$

C. $x \ge 0, \quad y \le 2$

D. $x \ge 0, \quad y \le -2$

4 Which of the following is a primitive of $\dfrac{4}{\sqrt{1-4x^2}}$?

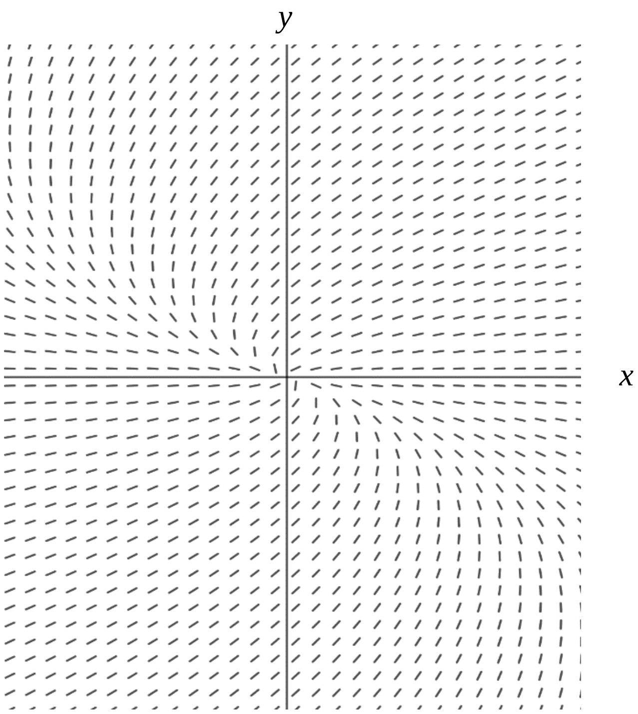
A. $2\sin^{-1}(2x)+C$

B. $2\sin^{-1}\left(\dfrac{x}{2}\right)+C$

C. $\sin^{-1}\dfrac{x}{2}+C$

D. $\sin^{-1}(2x)+C$

5 The slope field for a differential equation is shown below



Which of the following could be the differential equation represented?

A. $\dfrac{dy}{dx}=\dfrac{x}{x+y}$

B. $\dfrac{dy}{dx}=\dfrac{y}{x+y}$

C. $\dfrac{dy}{dx}=\dfrac{x}{x-y}$

D. $\dfrac{dy}{dx}=\dfrac{y}{x-y}$

6 Three candidates are running for election. There are 202 people who vote. The winner is the person who gets the most votes. What is the minimum number of votes needed for someone to win the election?

A. 101

B. 102

C. 67

D. 68

7 Which expression would be the correct answer to $\int \sin(5x)\sin(3x)\,dx$?

A. $\frac{1}{4}\cos(2x)+\frac{1}{16}\cos(8x)+C$

B. $\frac{1}{4}\cos(2x)-\frac{1}{16}\cos(8x)+C$

C. $\frac{1}{4}\sin(2x)+\frac{1}{16}\sin(8x)$ +C

D. $\frac{1}{4}\sin(2x)-\frac{1}{16}\sin(8x)+C$

8 A team of 7 is to be selected from 10 people and then sat around a circular table.
In how many ways can this be done?

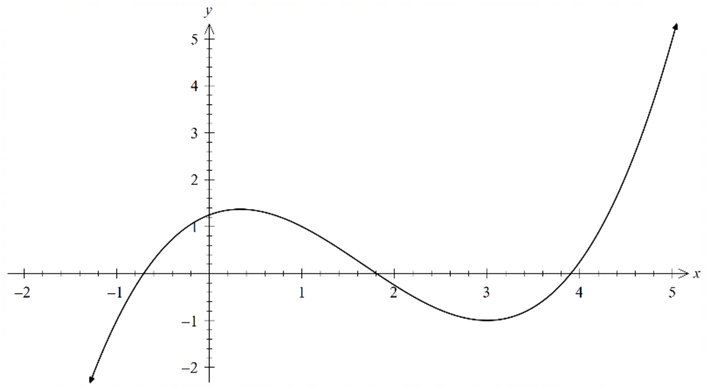
A. $\frac{10!}{7!3!}\times 7!$

B. $\frac{10!}{7!3!}\times\frac{1}{5!}$

C. $\frac{10!}{7\times 3!}$

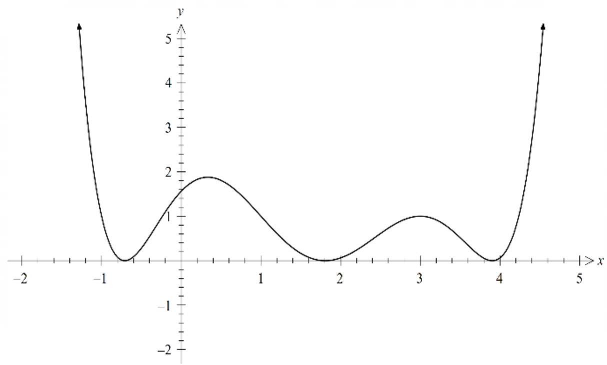
D. $\frac{10!}{7!3!}$

9 The graph of the function $y = f(x)$ is drawn below

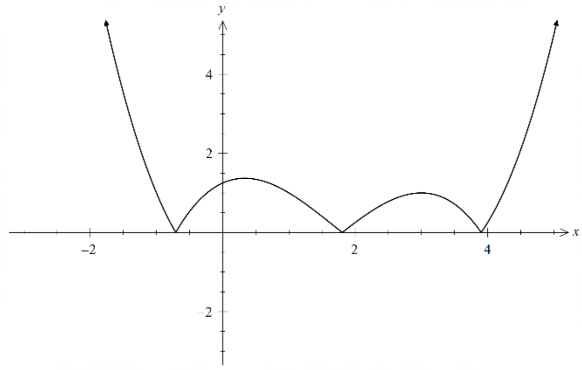


Which of the following represents the graph of $y = \left| f(|x|) \right|$?

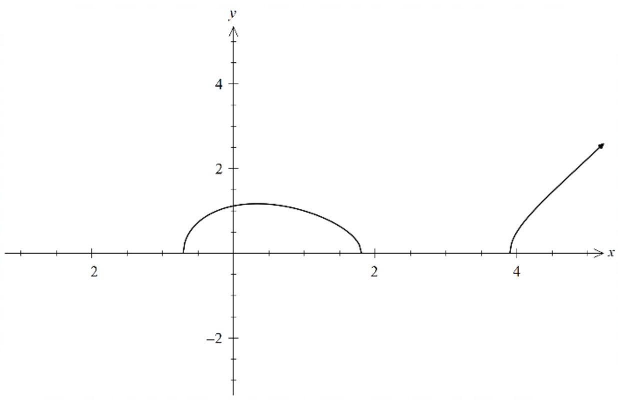
A



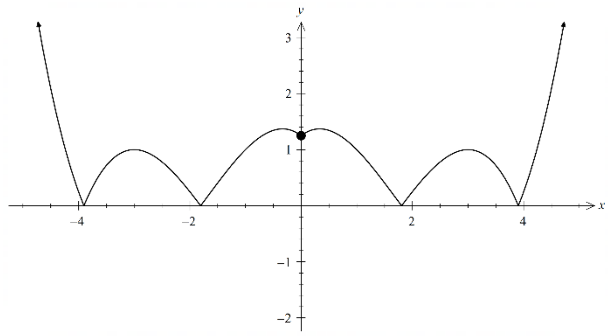
B



C



D



10 The parametric equations of a curve C are given by $x = \operatorname{cosec}^2 t$ and $y = \cos^2 t$.
What is the Cartesian equation of C ?

A. $y = 1 - \dfrac{1}{x}$

B. $y = 1 + \dfrac{1}{x}$

C. $xy = 1$

D. $y^2 = 1 - \dfrac{1}{x^2}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a separate writing booklet.
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) $P(x) = 2x^3 + 11x^2 + 12x - 9$

(i) Show that $P(-3) = 0$ **1**

$$P(x) = 2x^3 + 11x^2 + 12x - 9$$
$$P(-3) = 2(-3)^3 + 11(-3)^2 + 12(-3) - 9$$
$$= -54 + 99 - 36 - 9$$
$$= 0 \quad \checkmark$$

(ii) Show that $P'(-3) = 0$ **1**

$$P(x) = 2x^3 + 11x^2 + 12x - 9$$
$$P'(x) = 6x^2 + 22x + 12$$
$$P'(-3) = 6(-3)^2 + 22(-3) + 12$$
$$= 54 - 66 + 12 = 0 \quad \checkmark$$

(iii) Hence express $2x^3 + 11x^2 + 12x - 9$ as the product of 3 linear factors. **1**

$$P(x) = \left(x+3\right)^2\left(2x-1\right) \quad \checkmark$$

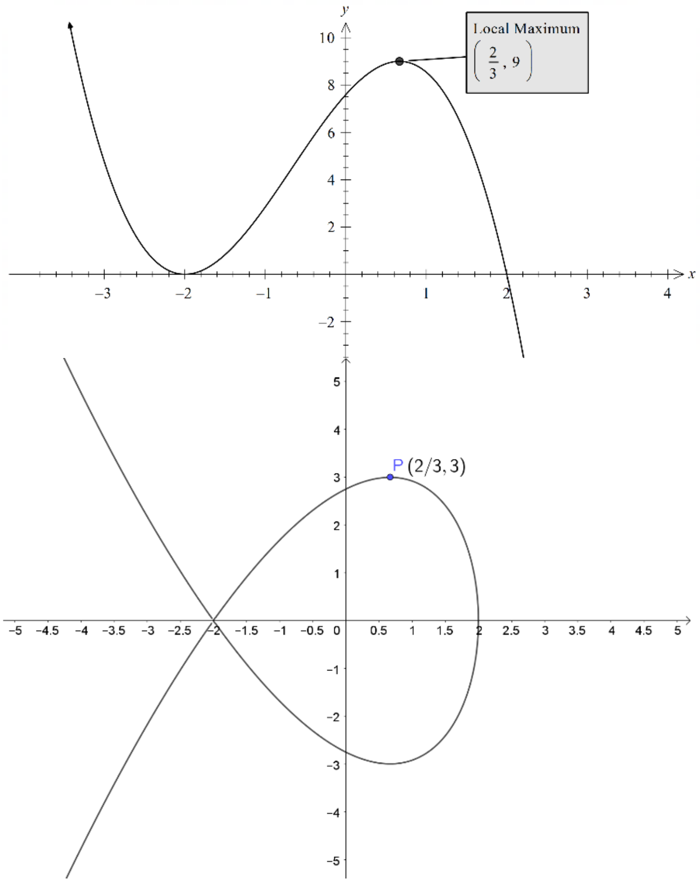
(b) For what value(s) of a are the two distinct vectors $\begin{pmatrix} a \\ 2a+6 \end{pmatrix}$ and $\begin{pmatrix} a+1 \\ -1 \end{pmatrix}$ perpendicular? **2**

To be perpendicular the dot product of the vectors must be 0. ✓

$$\begin{pmatrix} a \\ 2a+6 \end{pmatrix} . \begin{pmatrix} a+1 \\ -1 \end{pmatrix} = a^2 + a - 2a - 6$$
$$= a^2 - a - 6$$
$$a^2 - a - 6 = 0$$
$$\left(a-3\right)\left(a+2\right) = 0 \rightarrow a = 3, -2 \quad \checkmark$$

(c) The diagram shows the sketch of $y = f(x)$. Sketch the graph of $y^2 = f(x)$. **3**

Indicate the local maximum turning point on your diagram.



✓✓✓

(d) By letting $t = \tan\frac{x}{2}$, solve $4\sin x - 3\cos x = 3$, for $0 \le x \le 2\pi$ **3**

$$\text{Let } t = \tan\frac{x}{2} \to 4 \times \frac{2t}{1+t^2} - 3 \times \frac{1-t^2}{1+t^2} = 3 \quad ✓$$

$$\to 8t - 3 + 3t^2 = 3 + 3t^2$$

$$t = \frac{3}{4} \to x = 2\arctan\left(\frac{3}{4}\right) = 1.287\,(3dp) \quad ✓$$

Check $x = \pi \to 4\sin\pi - 3\cos\pi = 0 - -3 = 3$ ✓

$\therefore x = \pi$ is also a solution.

(e) Solve $2\sin^2 2x - 1 = 0$, over the domain $[0, \pi]$ finding x as a function of y. **2**

$$2\sin^2 2x - 1 = 0$$
$$\sin^2 2x = \frac{1}{2}$$
$$\sin 2x = \pm\frac{1}{\sqrt{2}} \quad 0 \le 2x \le 2\pi \quad ✓$$
$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8} \quad ✓$$

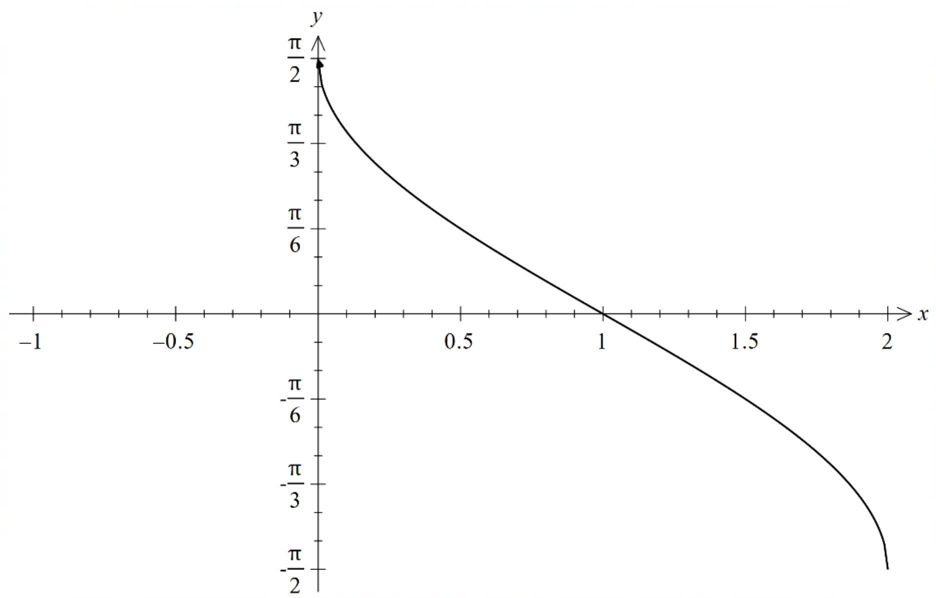
(f) A function is defined by $f(x) = \sin^{-1}(1-x)$.

(i) Find the domain of $f(x)$. **1**

$$-1 \le 1 - x \le 1$$
$$-2 \le -x \le 0$$
$$0 \le x \le 2 \quad ✓$$

(ii) Draw a neat sketch of $y = f(x)$ with correct range and x-intercept. **1**

✓



Question 12 (15marks)

(a) Use mathematical induction to prove that $n^3 + 2n$ is divisible by 3, for all positive integers *n*. **3**

Prove true for $n = 1$

$1^3 + 2\times 1 = 3$ which is divisible by 3 so true for $n = 1$ ☑

Assume true for $n = k$

$\frac{k^3 + 2k}{3} = M$ (where M is a positive integer)

$k^3 = 3M - 2k$

Prove true for $n = k + 1$

RTP that $(k+1)^3 + 2(k+1)$ is divisible by 3

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 3k^2 + 5k + 3 \\ &= 3M - 2k + 3k^2 + 5k + 3 \text{(from assumption)} \quad ☑ \\ &= 3(M + K + 1) \rightarrow (\text{divisible by } 3) \quad ☑\end{aligned}$$

true for $n = k + 1$, proved by mathematical induction

(b) Weather observations in the town of Dampville have established that the probability of rain on any given day is 0.8.

Observations are made for 100 consecutive days.

Let *X* be the random variable representing the number of rainy days.

(i) Find the expected value $E(X)$. **1**

$E(X) = np = 100\times 0.8 = 80$ days ✓

(ii) Find the standard deviation of *X*. **1**

$\sigma = \sqrt{npq} = \sqrt{100\times 0.8\times 0.2} = 4$ ✓

(iii) By considering a normal distribution find the approximate probability that $76 \le X \le 84$. **1**

This represents one standard deviation from the mean so

$P(76 \le X \le 84) \approx 0.68$ ✓

(c) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2x\,dx$ **3**

$$\int_0^{\frac{\pi}{2}} \cos^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 4x)\,dx \quad \checkmark$$

$$= \frac{1}{2}\left[x + \frac{\sin(4x)}{2}\right]_0^{\frac{\pi}{2}} \quad \checkmark$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin\left(4\times\frac{\pi}{2}\right)}{2} - 0\right)$$

$$= \frac{\pi}{4} \quad \checkmark$$

(d) Solve the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$, given that $y(-1) = 1$ **3**

Express your answer in the form $ay^3 + bx^2 + cy + d = 0$, where a,b,c,d are integers.

$$\frac{dy}{dx} = \frac{-x}{1+y^2}$$

$$dy\left(1+y^2\right) = -x\,dx$$

$$y + \frac{y^3}{3} = -\frac{x^2}{2} + C \text{ and } y(-1) = 1 \ \boxed{\checkmark}$$

$$1 + \frac{1^3}{3} = -\frac{(-1)^2}{2} + C \to C = \frac{11}{6} \ \boxed{\checkmark}$$

$$y + \frac{y^3}{3} = -\frac{x^2}{2} + \frac{11}{6} \to 2y^3 + 3x^2 + 6y - 11 = 0 \ \boxed{\checkmark}$$

(e) The roots of $x^3 - 2x^2 + 5x + 1 = 0$ are α, β and γ. Find the value of

(i) $5\alpha + 5\beta + 5\gamma$. **1**

$$5\alpha + 5\beta + 5\gamma = 5(\alpha + \beta + \gamma)$$

$$= 5 \times 2 = 10 \ \boxed{\checkmark}$$

(ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

$$\alpha\beta\gamma(\alpha + \beta + \gamma) = -1 \times 2 = -2 \ \boxed{\checkmark}$$

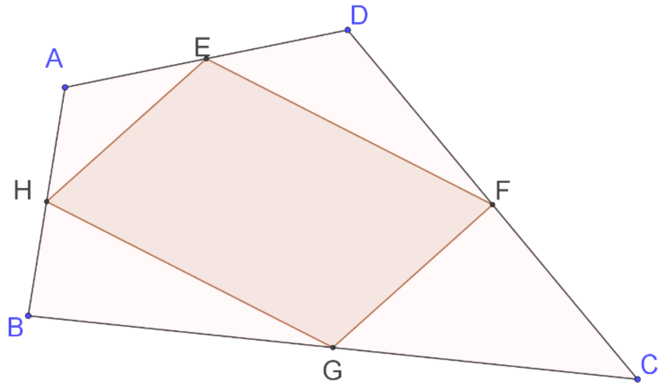
(iii) $\alpha^2 + \beta^2 + \gamma^2$. **1**

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= 2^2 - 2(5) = -6 \ \boxed{\checkmark}$$

Question 13 (15marks)

(a) Varignon's theorem tells us that if you join the midpoints of the sides of any quadrilateral, the resulting quadrilateral will be a parallelogram.



E, *F*, *G* and *H* are the midpoints of *AD*, *DC*, *CB* and *BA* respectively.
Let $\overrightarrow{BA} = \underset{\sim}{u}, \overrightarrow{AD} = \underset{\sim}{v}, \overrightarrow{DC} = \underset{\sim}{w}$ and $\overrightarrow{CB} = \underset{\sim}{z}$

(i) Explain why $\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w} + \underset{\sim}{z} = 0$ **1**

$\overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB} = \overrightarrow{BB} = 0$ ☑

(ii) Show that $\overrightarrow{HE} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$

$\overrightarrow{HA} + \overrightarrow{AE} = \overrightarrow{HE}$

$\overrightarrow{HE} = \frac{1}{2}\underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v}$ **1**

$\overrightarrow{HE} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$ ☑

(iii) Show that $\overrightarrow{GF} = -\frac{1}{2}(\underset{\sim}{w} + \underset{\sim}{z})$ **1**

$\overrightarrow{GF} = \overrightarrow{GC} + \overrightarrow{CF}$

$= -\frac{1}{2}\underset{\sim}{z} + -\frac{1}{2}\underset{\sim}{w}$

$= -\frac{1}{2}(\underset{\sim}{w} + \underset{\sim}{z})$ ☑

(iv) Use your result in (i) to explain why *HE* is parallel to *GF* **2**

$\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w} + \underset{\sim}{z} = 0$

$\underset{\sim}{u} + \underset{\sim}{v} = -(\underset{\sim}{w} + \underset{\sim}{z})$ ☑

$\overrightarrow{HE} = \overrightarrow{GF}$

Since the vectors are equal they are parallel. ☑

(b) Use the substitution $u = 3 + e^x$ to find the exact value of $\displaystyle\int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}}\,dx$ **3**

$u = 3 + e^x \to du = e^x dx$ and $u_1 = 3 + e^{\ln 6} = 9$ and $u_2 = 3 + e^0 = 4$ ☑

$$\int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}}\,dx = \int_4^9 \frac{du}{\sqrt{u}} \quad ✓$$

$$= 2\left[\sqrt{u}\right]_4^9$$

$$= 2\left(\sqrt{9} - \sqrt{4}\right) = 2 \quad ✓$$

(c) A population of monkeys on a small island is an example of logistic growth. The population of the monkeys P is given $P = \dfrac{50}{1+24e^{-2t}}$ and t is in years.

(i) Determine the carrying capacity of the population. **1**

As $t \to \infty, P \to \dfrac{50}{1+0}$ so the carrying capacity is 50 monkeys. ✓

(ii) Show that $\dfrac{dP}{dt} = \dfrac{1}{25}P(50-P)$. **4**

$$P = \frac{50}{1+24e^{-2t}} = 50\left(1+24e^{-2t}\right)^{-1}$$

$$\frac{dP}{dt} = -50\left(1+24e^{-2t}\right)^{-2} \times -48e^{-2t} \quad ✓$$

$$= \frac{2400e^{-2t}}{\left(1+24e^{-2t}\right)^2} \text{ and } P = \frac{50}{1+24e^{-2t}} \to 1+24e^{-2t} = \frac{50}{P} \quad ✓$$

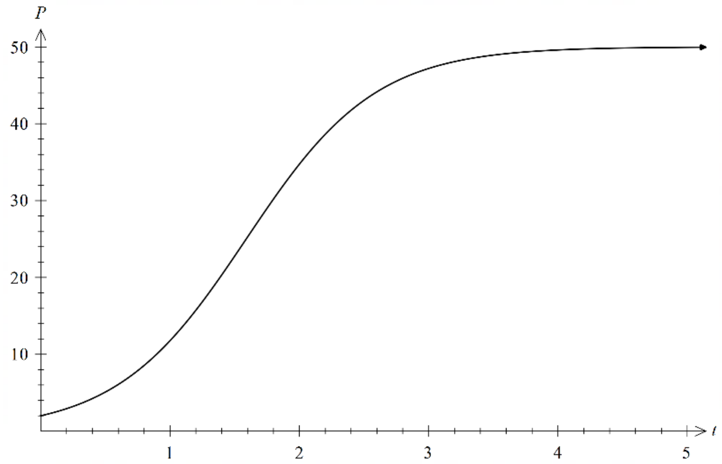
$$= \frac{24e^{-2t} \times 100}{\left(\frac{50}{P}\right)^2}$$

$$= \frac{100 \times \left(\frac{50}{P} - 1\right)}{\left(\frac{50}{P}\right)^2} \quad ✓$$

$$= 100 \times \left(\frac{50}{P} - 1\right) \times \frac{P^2}{2500}$$

$$= \frac{1}{25}P(50-P) \quad ✓$$

(iii) Draw a neat sketch of $P = \dfrac{50}{1+24e^{-2t}}$, taking care to consider the shape of the function at $t = 0$. Shape and Asymptote at P=50 **2**



✓ ✓ Stat at (0,2)

Question 14 (15 marks)

(a) Find the value of n if ${}^nC_2 + {}^nC_1 + {}^nC_0 = 172$ **3**

$${}^nC_2 + {}^nC_1 + {}^nC_0 = 172$$

$$\frac{n(n-1)}{2} + n + 1 = 172 \quad ✓$$

$$n^2 - n + 2n + 2 = 344$$

$$n^2 + n - 342 = 0 \quad ✓$$

$$(n-18)(n+19) = 0$$

$$n = 18, n \neq -19 ✓$$

(b) Let $f(x) = \dfrac{1}{\arcsin x}$. **3**

Find $f'(x)$ and the largest possible for which $f'(x)$ is defined.

$$f(x) = \frac{1}{\arcsin x} = (\arcsin x)^{-1}$$

$$f'(x) = -(\arcsin x)^{-2} \times \frac{1}{\sqrt{1-x^2}} \quad ✓$$

$$= \frac{-1}{(\arcsin x)^2 \sqrt{1-x^2}} \quad ✓$$

Both functions in the denominator have the same domain,but cannot be zero
So the domain is given by $-1 < x < 0$ or $0 < x < 1$ ✓ .

(c)

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + ... + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}$$ ✓

differentiate both sides

$$2n(1+x)^{2n-1} = \binom{2n}{1} + 2\binom{2n}{2}x + ... + (2n-1)\binom{2n}{2n-1}x^{2n-2} + 2n\binom{2n}{2n}x^{2n-1}$$ ✓

substitute $x = 1$

$$2n(2)^{2n-1} = \binom{2n}{1} + 2\binom{2n}{2} + ... + (2n-1)\binom{2n}{2n-1} + 2n\binom{2n}{2n}$$ ✓

$$n\left(2 \times 2^{2n-1}\right) = \binom{2n}{1} + 2\binom{2n}{2} + ... + (2n-1)\binom{2n}{2n-1} + 2n\binom{2n}{2n}$$ ✓

$$\binom{2n}{1} + 2\binom{2n}{2} + ... + (2n-1)\binom{2n}{2n-1} + 2n\binom{2n}{2n} = n \times 2^{2n}$$

(d) (i) Since $p = 0.23 < 0.5$, where p is the probability of a student studying Mathematics Extension 1, then the distribution is skewed to the right, i.e. positive skewness.

(ii)

Criteria	Marks
Provides correct solution	**4**
Correct value of z, or equivalent merit	**3**
Correct value of σ, or equivalent merit	**2**
Correct value of μ, or equivalent merit	**1**

Sample answer

$\mu = p = 0.23$	$\sigma^2 = \frac{p(1-p)}{n}$ $\sigma^2 = \frac{0.23 \times 0.77}{60}$ $\sigma^2 = 0.0029516..$ $\sigma = 0.054329...$	$z = \frac{x-\mu}{\sigma}$ $z = \frac{0.3-0.23}{0.054329}$ $z = 1.28844...$ $z = 1.29$

$P(z < 1.29) = 0.9015$

Therefore,

$$P(z > 1.29) = 1 - 0.9015$$
$$= 0.0985$$